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Lorentz covariance of heat conduction laws and a Lorentz-covariant heat conduction model

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ABSTRACT

Lorentz covariance is one of the two basic assumptions in relativity and it has considerable universality in nature. Lorentz covariance has been investigated widely in various fields including thermodynamics, but studies of heat conduction remain very limited. In this study, we demonstrate that several typical heat conduction laws, i.e. Fourier's law, the Cattaneo–Vernotte (CV) model, and Jeffery model, are not Lorentz-covariant. Thus, we propose a new heat conduction model that satisfies Lorentz covariance. Compared with the existing heat conduction laws, which are not Lorentz-covariant, this new Lorentz-covariant model can ensure that singularity and violation of the second law of thermodynamics do not apply in any inertial reference system. The CV model and the new model both predict thermal wave phenomena, but the CV model predicts a pure wave whereas the new model predicts a composite of translational motion and a wave. Therefore, the new Lorentz-covariant model makes the features of heat conduction more coherent.

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1. Introduction

Lorentz covariance is one of the two basic assumptions of special relativity and it is also considered to have considerable universality in nature. In fluid dynamics and other fields involving macroscopic statistics, Lorentz covariance has been investigated widely, especially with respect to the basic equations in these fields [1–4]. In equilibrium thermodynamics, fairly in-depth investigations have been performed for the Lorentz transformations of temperature, entropy, pressure, and other thermodynamic quantities [5–10]. In non-equilibrium thermodynamics, the classical Fourier's law of heat conduction is often used to describe classical heat conduction problems. However, the limitations of Fourier's law have been elucidated in recent years [11–19]. One of the major problems is that Fourier's law predicts an infinite speed of heat perturbation propagation, which apparently violates relativity [20]. In addition, Fourier's law of heat conduction cannot predict the supertransient and high heat flux processes well [11–15].

Several modified heat conduction models have been proposed to overcome these limitations. The Cattaneo–Vernotte (CV) model [21–22] is the main example, which predicts the hyperbolic heat conduction equation and wave-like transport in heat conduction processes called thermal waves. The CV model agrees well with many experiments and simulation data [12–13]. The Jeffrey model [12] can be considered as an extension of the CV model, which includes the influence of temperature relaxation. Tzou [23] proposed a single-phase-lagging heat conduction model, which can reduce to the CV model by making a first-order Taylor series approximation. Anisinov et al. [24] proposed a model for metals based on the interactions between

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electrons and phonons, and Guyer et al. [25] provided a model for pure phonon heat conduction. There have also been further modifications and improvements to these classical models. For example, Tzou [26] proposed a dual-phase-lagging model to include the influence of temperature lag in his single-phase-lagging model, while Coleman et al. [27] improved the changing rate of the heat energy in the CV model. Most of these models are linear and they predict limited speed in heat conduction processes, which can overcome the infinite speed problem in Fourier's law. In addition, some nonlinear models can address this problem, such as the thermomass theory [28–32] for heat conduction based on relativity and the mass-energy equation, as well as alternative approaches to the analysis of the diffusion equation [33–35]. These models are nonlinear, but they are closer to Fourier's law and more concise than some linear models. In particular, in alternative approaches to the analysis of the diffusion equation, the equation can be changed into Burger's equation [33–35], which means that some existing conclusions in mathematics can be used to analyze heat conduction problems.

These modified heat conduction models can overcome the infinite heat propagation speed problem in Fourier's law, which violates relativity, but this does not mean that they obey relativity. A limited speed of heat conduction does not equal the Lorentz covariance required by relativity. In essence, a general heat conduction process can be viewed as the result of electromagnetic interactions and the classical electromagnetic theory is Lorentz-covariant [36]. In addition, the thermomass theory of heat conduction is based on relativity and the mass-energy equation, and thus it also requires that the heat conduction is Lorentz-covariant. Therefore, it is necessary that heat conduction laws obey Lorentz covariance. Unfortunately, investigations of this issue have been very limited.

In this study, we first discuss the Lorentz covariance of typical existing heat conduction models and we show that they are not Lorentz-covariant. Next, we propose a new heat conduction law that satisfies Lorentz covariance. We also provide detailed discussions of some aspects of this new model, including its prerequisites and the Lorentz transformation of physical properties, as well as its advantages, physical meaning, and comparisons with other heat conduction models. Based on these discussions, we show that heat conduction laws should be Lorentz-covariant to ensure that singularity, such as unlimited thermal conductivity, and violation of the laws of thermodynamics, such as negative relaxation time or negative thermal conductivity, do not apply in any inertial reference system. Therefore, the Lorentz-covariant heat conduction model can make the theoretical system of heat conduction more coherent. Finally, we discuss the traveling wave solutions of the CV model and the new model, where we show that although they both predict thermal wave phenomena; the CV model predicts a pure wave whereas the new model predicts a composite of translational motion and a wave.

2. Lorentz covariance of typical heat conduction laws

2.1. Fourier's law of heat conduction

There are four main viewpoints regarding the Lorentz transformation of temperature and heat [5–10,37–41]: (1) Plank–Einstein's transformation $T' = \sqrt{1-\beta^2}T$ [6], (2) Ott's transformation $T' = \frac{T}{\sqrt{1-\beta^2}}$ [9], (3) Landberg's view $T' = T$ [10], and (4) Newburgh's view [37] that there is no universal Lorentz transformation of temperature. Plank–Einstein's transformation can be deduced by the thermodynamic method [38] and statistical mechanics method [39]. However, Ott [9] proposed another opposite Lorentz transformation of temperature in 1963, and many arguments and different views have appeared subsequently. Landberg [10] stated that temperature is Lorentz invariant, while Newburgh [37] suggested that there is no universal Lorentz transformation of temperature and that each Lorentz transformation has its own specific condition. In the present study, we do not discuss this problem with respect to the first three views as the difference between the transformations is only the constant coefficient of temperature, which has little influence on the results of the present study. As a case study, we consider Ott's Lorentz transformation of temperature $T' = \frac{T}{\sqrt{1-\beta^2}}$ and heat $Q' = \frac{Q}{\sqrt{1-\beta^2}}$. The Lorentz transformations of differential operators are $\frac{\partial}{\partial x'} = \gamma(\frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t})$, $\frac{\partial}{\partial t'} = \gamma(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})$. Q is the quantity of heat, $q = \frac{\partial Q}{\partial t}$ is the heat flow, and T is the temperature in a certain inertial reference system. Q', T', and q' are the corresponding physical quantities in another inertial reference system, respectively. u is the relative velocity between the two inertial reference systems, with $\beta = \frac{u}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. We can obtain

$$q' = \frac{\partial Q'}{\partial t'} = \gamma\left(\frac{\partial Q'}{\partial t} + u\frac{\partial Q'}{\partial x}\right) = q + u\frac{\partial Q}{\partial x}, \tag{1}$$

$$\frac{\partial T'}{\partial x'} = \frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t}. \tag{2}$$

For one space dimensional problems, Fourier's heat conduction law is $q = -\lambda\frac{\partial T}{\partial x}$. Its Lorentz transformation should have the same form $q' = -\lambda'\frac{\partial T'}{\partial x'}$. Substituting Eqs. (1) and (2) into this formula yields:

$$q + u\frac{\partial Q}{\partial x} = -\lambda'\left(\frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t}\right). \tag{3}$$

We note that Fourier's law only contains the heat flow and temperature gradient, but Eq. (3) contains the heat flow and temperature gradient as well as the rate of temperature change and the energy field gradient. Thus, it is difficult to guarantee that Fourier's law exhibits Lorentz covariance. For example, in inertial reference system ***1***, we can find a point that satisfies the following condition: $q+u\frac{\partial Q}{\partial x}\neq 0$. Then, we can find another inertial reference system ***2*** that satisfies the following condition: $\frac{\partial T}{\partial x}+\frac{u}{c^2}\frac{\partial T}{\partial t}=0$. The relative velocity ***u*** is less than ***c***. Since $\frac{\partial T}{\partial x}+\frac{u}{c^2}\frac{\partial T}{\partial t}=0$, then the thermal conductivity λ' in Eq. (3) is nonexistent, so Fourier's law is singular in inertial reference system ***2***. Therefore, Fourier's law of heat conduction is not Lorentz-covariant.

2.2. CV model

For one space dimensional problems, the CV model is written as

$$q+\tau\frac{\partial q}{\partial t}+\lambda\frac{\partial T}{\partial x}=0. \tag{4a}$$

Its Lorentz transformation should have the same form

$$q'+\tau'\frac{\partial q'}{\partial t'}+\lambda'\frac{\partial T'}{\partial x'}=0. \tag{4b}$$

Substituting Eqs. (1) and (2) into Eq. (4b) leads to

$$q+u\frac{\partial Q}{\partial x}+\tau'\gamma\left(\frac{\partial q}{\partial t}+2u\frac{\partial q}{\partial x}+u^2\frac{\partial^2 Q}{\partial x^2}\right)+\lambda'\left(\frac{\partial T}{\partial x}+\frac{u}{c^2}\frac{\partial T}{\partial t}\right)=0. \tag{5}$$

We note that the CV model only contains the temperature gradient, heat flow, and its rate of change, but Eq. (5) contains these terms and other differential terms. Therefore, it is difficult or impossible to guarantee that the CV model exhibits Lorentz covariance. We can also provide an example similar to the case above for Fourier's law, as follows. In inertial reference system ***1***, we can find a point where the right-hand side of Eq. (5) is nonzero and $\frac{\partial T}{\partial x}+\frac{u}{c^2}\frac{\partial T}{\partial t}=0$. Therefore, the thermal conductivity λ' in Eq. (5) is also nonexistent, and thus Fourier's law is singular in inertial reference system ***2***. Therefore, the CV model is also not Lorentz-covariant.

2.3. Jeffrey model

For one space dimensional problems, the Jeffrey model is expressed as

$$q+\tau_1\frac{\partial q}{\partial t}+\lambda\left(\frac{\partial T}{\partial x}+\tau_2\frac{\partial^2 T}{\partial x\partial t}\right)=0. \tag{6}$$

Its Lorentz transformation should have the following form

$$q'+\tau_1'\frac{\partial q'}{\partial t'}+\lambda'\left(\frac{\partial T'}{\partial x'}+\tau_2'\frac{\partial^2 T'}{\partial x'\partial t'}\right)=0. \tag{7}$$

Substituting Eqs. (1) and (2) leads to

$$q+u\frac{\partial Q}{\partial x}+\tau_1'\gamma\left(\frac{\partial q}{\partial t}+2u\frac{\partial q}{\partial x}+u^2\frac{\partial^2 Q}{\partial x^2}\right)+\lambda'\left(\frac{\partial T}{\partial x}+\frac{u}{c^2}\frac{\partial T}{\partial t}\right)+\lambda'\gamma\tau_2'\left[\frac{\partial^2 T}{\partial x\partial t}\left(1+\frac{u^2}{c^2}\right)+\frac{u}{c^2}\frac{\partial^2 T}{\partial t^2}+u\frac{\partial^2 T}{\partial x^2}\right]=0. \tag{8}$$

This is very similar to the previous situation and other differential terms do not appear in the original Jeffrey model. We can also see that the Jeffrey model is not Lorentz-covariant.

We know that Fourier's law and the CV model are not Lorentz-covariant, but this behavior can be eliminated in a special condition. Consider a steady problem without a heat source. Then, we have $\frac{\partial T}{\partial t}=0$, $\frac{\partial^2 Q}{\partial x^2}=\frac{\partial q}{\partial x}=0$, $\frac{\partial Q}{\partial x}=0$. Substituting these into Eqs. (3) and (5) makes Eq. (3) reduce to $q+\lambda'\frac{\partial T}{\partial t}=0$. Compared with Fourier's law in the initial inertial reference system, we have $\lambda'=\lambda$. I this case, the Lorentz transformation coefficients exist in Fourier's law and the thermal conductivity in Fourier's law is the same in different inertial reference systems, so Eq. (5) becomes $q+\tau'\gamma\frac{\partial q}{\partial t}+\lambda'\frac{\partial T}{\partial t}=0$. In contrast to the CV model in the initial inertial reference system, we obtain

$$\lambda'=\lambda, \tag{9}$$

$$\tau'=\frac{\tau}{\gamma}. \tag{10}$$

The physical properties of the Lorentz transformation of the CV model do exist in this special case and the transformations become Eqs. (9) and (10).

3. Lorentz-covariant heat conduction model

Our preconditions should be verified. First, we only consider the pure heat conduction problems by neglecting the effects of other processes, including chemical reaction, viscous dissipation, and mass transport. In addition, the medium should remain relatively static everywhere. Second, the problem should obey the continuous medium assumption and all of the differential operations are feasible. Third, the system should reach local equilibrium because the temperature is a state quantity based on statistics and it is defined for equilibrium.

In general, heat conduction processes can be regarded as the result of electromagnetic interactions. The diffusion of thermal energy can be considered as an electromagnetic interaction, and the diffusion of an electromagnetic interaction does not depend on whether a system is in local equilibrium. Therefore, the Lorentz-covariant model should contain terms similar to the electromagnetic equations to describe the electromagnetic interactions.

Thus, we can write a new Lorentz-covariant heat conduction model

$$q+\lambda\frac{\partial T}{\partial x}+a\frac{\partial Q}{\partial x}+b\frac{\partial T}{\partial t}+l\frac{\partial^2 Q}{\partial^2 t}-lc^2\frac{\partial^2 Q}{\partial^2 x}=0. \tag{11}$$

In the formula above, Q is the total heat flowing through x from time *0* to time *t*. $a\frac{\partial Q}{\partial x}$ and $b\frac{\partial T}{\partial t}$ are the symmetrical terms of q and $\lambda\frac{\partial T}{\partial x}$ for the symmetry in time and space, respectively, which are needed by the theory of relativity. $l\frac{\partial^2 Q}{\partial^2 t}$ and $lc^2\frac{\partial^2 Q}{\partial^2 x}$ are both wave terms, which reflect the thermal signal response and they are also the electromagnetic interaction terms in heat conduction processes. lc^2 is very large, which means that the electromagnetic interaction travels very rapidly in heat conduction processes and this interaction will reach the entire field in a very short time. Therefore, the influence due to the traveling of the electromagnetic interaction will disappear very rapidly. If the coefficients *a, b*, and *l* are all zero, Eq. (11) will reduce to Fourier's heat conduction law. For the steady problems without a heat source, we have $\frac{\partial T}{\partial t}=0$, $\frac{\partial^2 Q}{\partial x^2}=\frac{\partial q}{\partial x}=0$, $\frac{\partial Q}{\partial x}=0$, $\frac{\partial^2 Q}{\partial^2 t}=\frac{\partial q}{\partial t}=0$, and substituting them into Eq. (11) leads to $q+\lambda\frac{\partial T}{\partial t}=0$. Therefore, for the problem with *a, b*, and *l* zero coefficients, and the steady problems without a heat source, Eq. (11) is consistent with Fourier's heat conduction law. In physics, for the steady problems without a heat source, the parameters of the entire temperature field will be constant because "diffusion" and "relaxation" do not exist naturally. In general, the Fourier's heat conduction law agrees well with experimental results, and thus Eq. (11) can be reduced to Fourier's heat conduction law. If the coefficients *a, b*, and $\frac{\partial^2 Q}{\partial^2 x}$ are zero, we have the CV model $q+\lambda\frac{\partial T}{\partial x}+l\frac{\partial^2 Q}{\partial^2 t}=0$. Therefore, *l* becomes the relaxation time. The CV model also agrees well with many experiments and simulation data [12–13]. Thus, it is also reasonable to assume that Eq. (11) can be reduced to the CV model.

In general, most of the heat conduction processes obey Fourier's law because after a sufficient time, the influence of the thermal signal terms becomes negligible and thus the coefficient *l* should be very small in these terms. In addition, the coefficient *l* in $l\frac{\partial^2 Q}{\partial^2 t}$ is simply the relaxation time in the CV model. The relaxation time for matter is generally in the order of $ps\sim fs$. Next, we verify whether Eq. (11) is Lorentz-covariant.

The Lorentz transformations of differential operators are $\frac{\partial}{\partial x'}=\gamma(\frac{\partial}{\partial x}+\frac{u}{c^2}\frac{\partial}{\partial t})$, $\frac{\partial}{\partial t'}=\gamma(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x})$, and thus we obtain

$$q'=\frac{\partial Q'}{\partial t'}=\gamma\left(\frac{\partial Q'}{\partial t}+u\frac{\partial Q'}{\partial x}\right)=\frac{\partial Q}{\partial t}+u\frac{\partial Q}{\partial x}=q+u\frac{\partial Q}{\partial x}, \tag{12}$$

$$\frac{\partial T'}{\partial t'}=\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}, \tag{13}$$

$$\frac{\partial T'}{\partial x'}=\frac{\partial T}{\partial x}+\frac{u}{c^2}\frac{\partial T}{\partial t}, \tag{14}$$

$$\frac{\partial Q'}{\partial x'}=\frac{\partial Q}{\partial x}+\frac{u}{c^2}\frac{\partial Q}{\partial t}=\frac{\partial Q}{\partial x}+\frac{u}{c^2}q. \tag{15}$$

The Lorentz transformation of Eq. (11) should have the following form

$$q'+\lambda'\frac{\partial T'}{\partial x'}+a'\frac{\partial Q'}{\partial x'}+b\frac{\partial T'}{\partial t'}+l'\frac{\partial^2 Q'}{\partial^2 t'}-l'c^2\frac{\partial^2 Q'}{\partial^2 x'}=0. \tag{16}$$

Substituting Eqs. (12–15) into Eq. (16) leads to

$$\left(1+\frac{a'u}{c^2}\right)q+(\lambda'+b'u)\frac{\partial T}{\partial x}+(a'+u)\frac{\partial Q}{\partial x}+\left(b'+\frac{\lambda' u}{c^2}\right)\frac{\partial T}{\partial t}+\gamma l'\frac{\partial^2 Q}{\partial^2 t}-\gamma l'c^2\frac{\partial^2 Q}{\partial^2 x}=0. \tag{17}$$

By comparing the coefficients of Eqs. (17) and (11), where each corresponding differential term should be proportional, we have the following equations

$$u+a'=a+a\frac{a'u}{c^2}, \tag{18a}$$

$$\lambda'+b'u=\lambda+\frac{a'u}{c^2}\lambda, \tag{18b}$$

$$b' + \frac{\lambda' u}{c^2} = b + b\frac{a' u}{c^2}, \tag{18c}$$

$$\gamma l' = l\left(1 + \frac{a' u}{c^2}\right). \tag{18d}$$

These four equations are linearly independent and there are four unknown parameters. The four equations are solvable, so we can obtain

$$a' = \frac{a - u}{1 - \frac{au}{c^2}}, \tag{19a}$$

$$b' = \frac{bc^2 - u\lambda}{c^2 - au}, \tag{19b}$$

$$\lambda' = \frac{c^2(\lambda - bu)}{c^2 - au}, \tag{19c}$$

$$l' = \frac{l}{\gamma}\frac{c^2 - u^2}{c^2 - au}. \tag{19d}$$

Therefore, we can conclude that Eq. (11) is Lorentz-covariant. After the Lorentz transformation, the physical property coefficients of Eq. (11) become Eqs. (19a–d). Note that all of the denominators contain $c^2 - au$, where u is the relative velocity between the two inertial reference systems, which should be less than the speed of light c. Then, let us consider the range of the coefficient a. In Eq. (11), if we only consider the influence of a, then we have $q + a\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial t} + a\frac{\partial Q}{\partial x} = 0$. The general solution of this equation is $x = -at + C$, where C is a constant. Therefore, we have $\frac{dx}{\partial t} = -a$. Then, we can consider that the coefficient a is the rate of the diffusion of energy. a should be less than the speed of light c, so the denominator is

$$c^2 - au > 0. \tag{20}$$

Therefore, we can guarantee that each physical property coefficient is finite.

Let us turn our attention to the physical property coefficients when u is far less than c. We also have $a < c$, and thus au is far less than c^2. Then, we can have $1 - \frac{au}{c^2} \approx 1$, $c^2 - au \approx c^2$, $\gamma \approx 1$, $c^2 - u^2 \approx c^2$. Substituting them into Eqs. (19a–d) yields

$$a' \approx a - u, \tag{21a}$$

$$b' \approx b, \tag{21b}$$

$$\lambda' \approx \lambda - bu, \tag{21c}$$

$$l' \approx l. \tag{21d}$$

Provided that u is less than c, the heat flow produced by the temperature gradient should be in the opposite direction to the temperature gradient's direction. We have $u < c$ and $\lambda - bu > 0$. Therefore,

$$b \le \frac{\lambda}{c}. \tag{22}$$

Then, we have

$$bu \le \frac{\lambda u}{c}. \tag{23}$$

Finally, we have $bu < <\lambda$. Substituting this into Eq. (14) leads to

$$\lambda' \approx \lambda. \tag{24}$$

In summary, for the case where u is far less than c, the physical property coefficients b, l, λ are very close to the physical property coefficients of the stationary reference system. In other words, the relativistic effects are generally reflected in the physical property coefficient a, and the relativistic effects will only be obvious when the coefficient a is close to the relative velocity u.

4. Advantages of the Lorentz-covariant model

For the new heat conduction model, Eq. (11), we have Eqs. (20) and (22), which can guarantee that the thermal conductivity $\lambda' = \frac{c^2(\lambda - bu)}{c^2 - au}$ is always positive in any inertial reference system. Therefore, the new heat conduction model guarantees the second law of the thermodynamics statement given by Clausius. However, Fourier's heat conduction law cannot guarantee this in any inertial reference system. Since $c^2 - au > 0$ and $c > u$, then we also have $l' = \frac{l}{\gamma}\frac{c^2 - u^2}{c^2 - au} > 0$, which means that the new heat conduction model can also guarantee that the relaxation time is positive in any inertial reference system, whereas this cannot be guaranteed by the CV model. Next, we discuss the problems of thermal conductivity and the relaxation time.

From Eqs. (1) and (2), we can obtain the apparent thermal conductivity of Fourier's heat conduction law in another inertial reference system

$$\lambda' = -\frac{q'}{\frac{\partial T'}{\partial x'}} = -\frac{q + u\frac{\partial Q}{\partial x}}{\frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t}}. \tag{25}$$

For Eq. (25), we can only determine $q\frac{\partial T}{\partial x} < 0$, whereas we cannot determine whether Eq. (25) is positive. Even the case where $\frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t} = 0$ will cause singularity.

From Eq. (5), we have the relaxation time of the CV model

$$\tau' = -\frac{\lambda'\left(\frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t}\right) + q + u\frac{\partial Q}{\partial x}}{\gamma\left(\frac{\partial q}{\partial t} + 2u\frac{\partial q}{\partial x} + u^2\frac{\partial^2 Q}{\partial x^2}\right)}. \tag{26}$$

For Eq. (26), we can only ascertain that

$$\tau = \frac{\lambda\frac{\partial T}{\partial x} - q}{\frac{\partial q}{\partial t}} > 0. \tag{27}$$

However, many differential terms do not appear in Eq. (27), so we cannot determine whether Eq. (27) is positive and the singularity caused by $\frac{\partial q}{\partial t} + 2u\frac{\partial q}{\partial x} + u^2\frac{\partial^2 Q}{\partial x^2} = 0$ will also appear. A negative relaxation time means that a heat flow response appears before a temperature perturbation, which obviously violates the second law of thermodynamics. Similarly, the same situation will occur in the Jeffrey model. Therefore, Fourier's heat conduction law, the CV model, and the Jeffrey model are all not Lorentz-covariant, and they violate the second law of thermodynamics in certain inertial reference systems. In fact, this reflects an important and significance feature of a Lorentz-covariant heat model, i.e., it guarantees that the heat conduction law will not violate the second law of thermodynamics in any inertial reference system. A Lorentz-covariant heat conduction law has the same form in any inertial reference system and it must obey the second law of thermodynamics in any inertial reference system. By contrast, the heat conduction laws that are not Lorentz-covariant may have different forms in various inertial reference systems, and we cannot ensure that they obey the second law of thermodynamics in any inertial reference system.

Next, we provide an example that compares the models described above. For Fourier's law, we consider a heat conduction process without a heat source. The initial conditions are taken as $T(x, 0) = T_0(1 + sin\frac{\pi x}{l})$. The boundary conditions are taken as $T = T_0$ ($x = 0$ or $x = l$). For this problem, we can obtain the classical solution as $T(x, t) = T_0(1 + e^{-\frac{\lambda\pi^2 t}{\rho c_V l^2}} sin\frac{\pi x}{l})$. Let u satisfy $q + u\frac{\partial Q}{\partial x} \neq 0$ in Eq. (25), and then we consider the denominator $\frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t}$. If we substitute the classical solution into the denominator, we obtain $\frac{\partial T}{\partial x} + \frac{u}{c^2}\frac{\partial T}{\partial t} = T_0\frac{\pi}{l}\cos\frac{\pi x}{l}e^{-\frac{\lambda\pi^2 t}{\rho c_V l^2}}(1 - \frac{u}{c^2}\frac{\lambda\pi}{\rho c_V l}\tan\frac{\pi x}{l})$. The range of the tangent function is the whole real number field, so the denominator must be zero somewhere. $q + u\frac{\partial Q}{\partial x} \neq 0$, and thus Fourier's law does cause singularity in some inertial reference systems. For the new model and the CV model, more initial conditions are needed to obtain a definite solution. We can set these initial conditions so they lead to the same solution as Fourier's law. Then, we can obtain the thermal conductivity distributions of the new model and the CV model, and perform a comparison of these models, as shown in Fig. 1. In this case, the physical properties satisfy $u = 0.5c$, $\frac{\pi\lambda}{2c\rho c_V l^2} = 1$, $\frac{2\pi\lambda\tau\gamma}{\rho c_V l^2} = 1$, $\frac{bc}{2\lambda} = 1$, and $\frac{a}{c} = 1$. From Fig. 1, we can find the singularity at $\frac{x}{l} = 0.25$ as well as the negative thermal conductivity caused by the CV model and Fourier's law. However, the thermal conductivity is a positive constant for the new model.

5. Traveling wave solution

The CV model and the Lorentz-covariant model predict the wave transport of thermal energy, which is called a thermal wave. For the CV model and Eq. (11), we can use the method of separating the variables directly to obtain the solution and we can also use the method of characteristics to determine the traveling wave solution. Next, we use the method of characteristics to obtain the traveling wave solution for one space dimensional constant physical properties problems and we discuss the thermal waves.

5.1. Traveling wave solution of the CV model

First, we have the energy conservation equation

$$-\frac{\partial q}{\partial x} = \rho c_V\frac{\partial T}{\partial t}, \tag{28}$$

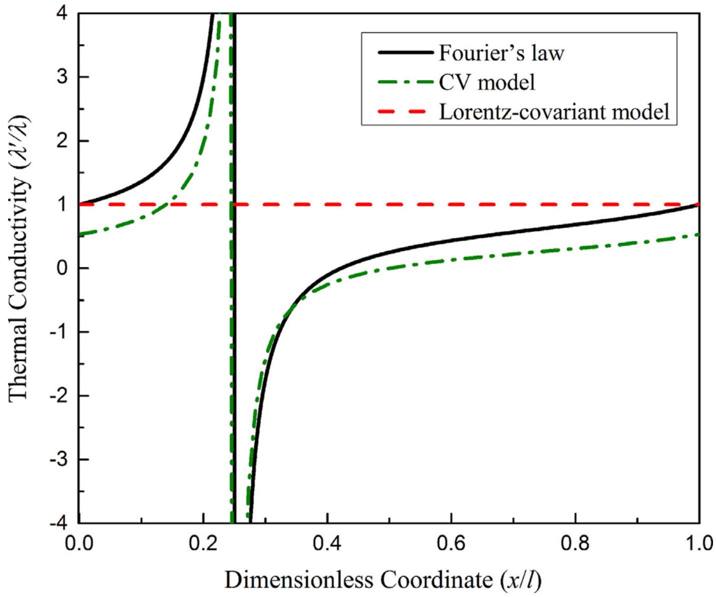

Fig. 1. Thermal conductivity distribution in the inertial reference system.

where ρ is the mass density and c_V is the specific heat. From Eqs. (4a) and (28), we obtain

$$\frac{\lambda}{\tau\rho c_V}\frac{\partial^2 T}{\partial^2 x} = \frac{1}{\tau}\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial^2 t}. \tag{29}$$

Then, we can use the method of characteristics. Consider a variable transformation

$$\frac{\lambda}{\tau\rho c_V} = v^2, \quad \xi = x - vt, \quad \eta = x + vt. \tag{30}$$

Substituting Eq. (30) into Eq. (29) leads to

$$4v\tau\frac{\partial^2 T}{\partial\eta\partial\xi} = \frac{\partial T}{\partial\eta} - \frac{\partial T}{\partial\xi}. \tag{31}$$

Then, we perform the separation of variables $T = f(\eta)g(\xi)$. Substituting this into Eq. (31) gives

$$4v\tau f'g' = f'g - g'f, \tag{32}$$

$$4v\tau = \frac{g}{g'} - \frac{f}{f'}. \tag{33}$$

Therefore, we have the eigenvalues of Eq. (33)

$$\frac{f}{f'} = \lambda_n, \quad \frac{g}{g'} = \lambda_n + 4v\tau. \tag{34}$$

By solving the two ordinary differential equations above, we obtain

$$f = C_1 e^{\frac{\eta}{\lambda_n}}, g = C_2 e^{\frac{\xi}{\lambda_n + 4v\tau}}, \tag{35}$$

$$T_n = C_n e^{\frac{\xi}{\lambda_n + 4v\tau} + \frac{\eta}{\lambda_n}}, \tag{36}$$

$$T = \sum C_n e^{\frac{\xi}{\lambda_n + 4v\tau} + \frac{\eta}{\lambda_n}}. \tag{37}$$

Substituting Eq. (30) into Eq. (37) leads to

$$T = \sum C_n e^{\frac{x+vt}{\lambda_n + 4v\tau} + \frac{x-vt}{\lambda_n}}. \tag{38}$$

The eigenvalues λ_n and the coefficient C_n are determined by the boundary and initial conditions. From Eq. (38), we find that the solution for the temperature does not have a general solution in the form of the wave equation $u = f_1(x - vt) + f_2(x + vt)$. However, we find that for every temperature's wavelets that belong to every eigenvalue, if the logarithm $\ln T_n$ of the wavelet T_n is taken, then it has the wave form of $u = f_1(x - vt) + f_2(x + vt)$. This indicates that the assumed thermal wave does not mean that the temperature or internal energy has a form that belongs to a solution of the wave equation, but instead some physical quantities that can be expressed as $C\ln T$ have the wave solution form (e.g., entropy).

5.2. Traveling wave solution of the Lorentz-covariant model neglecting a and b

In the case where the coefficients a and b are quite small and they can be neglected, Eq. (11), reduces to

$$q + l\frac{\partial^2 Q}{\partial t^2} + \lambda\frac{\partial T}{\partial x} - lc^2\frac{\partial^2 Q}{\partial x^2} = 0. \tag{39}$$

Eq. (28) is also the energy conservation equation. From Eqs. (39) and (28), we have

$$\frac{\lambda + \rho c_V lc^2}{l\rho c_V}\frac{\partial^2 T}{\partial^2 x} = \frac{1}{l}\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial^2 t}. \tag{40}$$

We note that Eq. (40) has the same form as Eq. (29), except that the coefficient $\frac{\partial^2 T}{\partial^2 x}$ is $\frac{\lambda+\rho c_V lc^2}{l\rho c_V}$. This is equivalent to the velocity of the wave being $\frac{\lambda+\rho c_V lc^2}{l\rho c_V} = v_2^2$. The rest of the solution process is the same as that for the CV model and the solution yields

$$T_2 = \sum C_n e^{\frac{x+v_2 t}{\lambda_n+4v\tau} + \frac{x-v_2 t}{\lambda_n}}. \tag{41}$$

Therefore, if the coefficients a and b are neglected, the solution of Eq. (11) has the same form as the CV model, except for the velocity of the wave.

5.3. Traveling wave solution of the Lorentz-covariant model neglecting b

If the coefficient b is quite small and it can be neglected, then Eq. (11) reduces to

$$q + l\frac{\partial^2 Q}{\partial t^2} + \lambda\frac{\partial T}{\partial x} + a\frac{\partial Q}{\partial x} - lc^2\frac{\partial^2 Q}{\partial x^2} = 0. \tag{42}$$

From Eqs. (42) and (28), we obtain

$$\frac{\lambda + \rho c_V lc^2}{l\rho c_V}\frac{\partial^2 T}{\partial^2 x} = \frac{1}{l}\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial^2 t} + \frac{a}{l}\frac{\partial T}{\partial x}. \tag{43}$$

Then, we use the method of characteristics again. We consider the variable transformations $\frac{\lambda+\rho c_V lc^2}{l\rho c_V} = v_2^2$, $\xi = x - v_2 t$, $\eta = x + v_2 t$ and by substituting them into Eq. (43), we have

$$4v_2^2 l\frac{\partial^2 T}{\partial\eta\partial\xi} = (a + v_2)\frac{\partial T}{\partial\eta} + (a - v_2)\frac{\partial T}{\partial\xi}. \tag{44}$$

Then, we perform a separation of variables $T = f(\eta)g(\xi)$, and after substitution, Eq. (44) leads to

$$4v_2^2 l f'g' = (a + v_2)f'g + (a - v_2)fg', \tag{45}$$

$$4v_2^2 l = (a + v_2)\frac{g}{g'} + (a - v_2)\frac{f}{f'}. \tag{46}$$

We also have the eigenvalues of Eq. (46)

$$\frac{f}{f'} = \lambda_n, \quad \frac{g}{g'} = \frac{4v_2^2 l + (v_2 - a)\lambda_n}{a + v_2}. \tag{47}$$

By solving the two ordinary differential equations above, we can obtain

$$f = C_1 e^{\frac{\eta}{\lambda_n}}, g = C_2 e^{\frac{\xi(a+v_2)}{4v_2^2 l + (v_2 - a)\lambda_n}}, \tag{48}$$

$$T_n = C_n e^{\frac{\eta}{\lambda_n} + \frac{\xi(a+v_2)}{4v_2^2 l + (v_2-a)\lambda_n}}. \tag{49}$$

Substituting $\xi = x - v_2 t$, $\eta = x + v_2 t$ into Eq. (49) gives

$$T_n = C_n e^{\frac{x+v_2 t}{\lambda_n} + \frac{(x+v_2 t)(a+v_2)}{4v_2^2 l + (v_2 - a)\lambda_n}}, \tag{50}$$

$$T = \sum C_n e^{\frac{x+v_2 t}{\lambda_n} + \frac{(x+v_2 t)(a+v_2)}{4v_2^2 l + (v_2 - a)\lambda_n}}. \tag{51}$$

The eigenvalues λ_n and the coefficients C_n are determined by the boundary and initial conditions. We note that Eq. (51) has the same form as the solution of the CV model, where their forms are both

$$T = \sum C_n e^{A_n(x+vt) + B_n(x-vt)}. \tag{52}$$

5.4. Traveling wave solution of the Lorentz-covariant model

From Eq. (11) and the energy conservation Eq. (28), we obtain

$$\frac{\lambda + c_p l c^2}{l c_p}\frac{\partial^2 T}{\partial^2 x} = \frac{1}{l}\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial^2 t} + \frac{a}{l}\frac{\partial T}{\partial x} - \frac{b}{c_p l}\frac{\partial^2 T}{\partial x \partial t}. \tag{53}$$

Then, we can still use the method of characteristics. Using variable transformations $\frac{\lambda+\rho c_V l c^2}{l\rho c_V} = v_2^2$, $\frac{b}{c_p l} = m$, we obtain

$$\xi = x + \frac{m - \sqrt{m^2+4v^2}}{2}t, \quad \eta = x + \frac{m + \sqrt{m^2+4v^2}}{2}t. \tag{54}$$

Substituting them into Eq. (53) leads to

$$(4v^2+m^2)l\frac{\partial^2 T}{\partial\eta\partial\xi} = \left(a + \frac{m+\sqrt{m^2+4v^2}}{2}\right)\frac{\partial T}{\partial \eta} + \left(a + \frac{m-\sqrt{m^2+4v^2}}{2}\right)\frac{\partial T}{\partial \xi}. \tag{55}$$

Then, we perform a separation of variables $T = f(\eta)g(\xi)$, and substituting them into Eq. (55) gives

$$(4v^2+m^2)l = \left(a + \frac{m+\sqrt{m^2+4v^2}}{2}\right)\frac{g}{g'} + \left(a + \frac{m-\sqrt{m^2+4v^2}}{2}\right)\frac{f}{f'}. \tag{56}$$

We have the eigenvalues of Eq. (56)

$$\frac{f}{f'} = \lambda_n, \quad \frac{g}{g'} = \frac{(4v^2+m^2)l - (a + \frac{m-\sqrt{m^2+4v^2}}{2})\lambda_n}{\left(a + \frac{m+\sqrt{m^2+4v^2}}{2}\right)}. \tag{57}$$

By solving the two ordinary differential equations above, we obtain

$$f = C_1 e^{\frac{\eta}{\lambda_n}}, \quad g = C_2 e^{\frac{\xi\left(a+\frac{m+\sqrt{m^2+4v^2}}{2}\right)}{(4v^2+m^2)l-(a+\frac{m-\sqrt{m^2+4v^2}}{2})\lambda_n}}, \tag{58}$$

$$T_n = C_n \exp\left[\frac{\eta}{\lambda_n} + \frac{\xi\left(a + \frac{m+\sqrt{m^2+4v^2}}{2}\right)}{(4v^2+m^2)l - \left(a + \frac{m-\sqrt{m^2+4v^2}}{2}\right)\lambda_n}\right], \tag{59}$$

$$T = \sum C_n \exp\left[\frac{x + \frac{m+\sqrt{m^2+4v^2}}{2}t}{\lambda_n} + \frac{\left(x + \frac{m-\sqrt{m^2+4v^2}}{2}t\right)\left(a + \frac{m+\sqrt{m^2+4v^2}}{2}\right)}{(4v^2+m^2)l - \left(a + \frac{m-\sqrt{m^2+4v^2}}{2}\right)\lambda_n}\right]. \tag{60}$$

The eigenvalues λ_n and the coefficients C_n are also determined by the boundary and initial conditions. The form of Eq. (60) is very similar to that of Eqs. (38), (41), and (51) at first glance, but they are actually not identical. Eqs. (38), (41), and (51) all have the form of $T = \sum C_n e^{A_n(x+vt)+B_n(x-vt)}$, and their wave characteristics are also identical. When the wavelet T_n is taken as the logarithm, we have $\ln T_n$ in the form of $u = f_1(x - vt) + f_2(x + vt)$, which is the strict solution form of the wave equation. However, the form of Eq. (60) is

$$T = \sum C_n e^{A_n(x+ut+vt)+B_n(x+ut-vt)}. \tag{61}$$

When the wavelet T_n is taken as the logarithm, we have $\ln T_n$ in the form of $f_1(x+ut+vt) + f_2(x+ut-vt)$, which obviously is not the strict solution form of the wave equation. There are wave terms as well as a translational term ut, which means that $\ln T_n$ in Eq. (62) is not a pure wave, but instead it is a composite of translational motion and a wave. This is the essential difference between Eq. (60) and Eqs. (38), (41), and (51).

In summary, for one space dimensional constant physical property problems, when some terms in Eq. (11) are neglected (i.e., $a = b = 0$ or $a = 0$), the solutions of the CV model and the approximate equations are strict solution forms of the wave equation. However, the exact solution of Eq. (11) is not a strict solution form of the wave equation. The solution is not a pure wave but instead it is a composite of translational motion and a wave. This distinction can be observed if the effect of the initial condition is symmetrical. For the CV model, the domain of determinacy, which belongs to coordinate x and time from 0 to t, is $[x - vt, x + vt]$, and the effect of the initial condition is symmetrical. However, for Eq. (11), the domain of determinacy, which belongs to the coordinate x and time from 0 to t, is $[x - vt + ut, x + vt + ut]$, and the effect of the initial condition is not symmetrical when the translational velocity u is nonzero.

6. Conclusions

(1) In this study, we demonstrated that several typical heat conduction models, such as Fourier's law, the CV model, and the Jeffrey model, are not Lorentz-covariant, which leads to singularity and violation of the second law of thermodynamics in inertial reference systems.

(2) Thus, we proposed a new heat conduction model, i.e., Eq. (11), which is Lorentz-covariant. We also provided detailed discussions of the features of this new model, including its prerequisites and the Lorentz transformation of physical properties, as well as its advantages, physical meaning, and comparisons with other heat conduction models.
(3) Compared with the existing heat conduction models, which are not Lorentz-covariant, this new Lorentz-covariant model ensures that singularity and violation of the laws of thermodynamics do not apply in any inertial reference system. This is certainly the main advantage of the new model.
(4) The new heat conduction model and the CV model both predict thermal wave phenomena, but the CV model characterizes a pure wave whereas the new model predicts a composite of translational motion and a wave.
(5) However, the Lorentz-covariance does not automatically imply that the new Lorentz-covariant model adequately describes physical reality without having sufficient experimental data for its verification. This new model will reduce to Fourier's law and the CV model in some special cases. In general, Fourier's law can address classical normal heat conduction problems and the CV model can explain the supertransient heat conduction, while they also agree well with some experiments and simulations. The new model can be considered as a Lorentz-covariant improvement of the CV model and Fourier's law. Some terms of Eq. (8) can illustrate its speculative nature, as shown by the CV model and Fourier's law in a stationary reference system, such as the linear relationship between the temperature gradient and heat flux. Other Lorentz-covariant correction terms are proposed to ensure that singularity and violation of the laws of thermodynamics do not apply in any other inertial reference system.

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References

[1] A.H. Taub, General relativistic variational principle for perfect fluids, Phys. Rev. 94 (6) (1954) 1468.
[2] J.R. Ray, Lagrangian density for perfect fluids in general relativity, J. Math. Phys. 13 (10) (1972) 1451.
[3] J.B.F. Schutz, Perfect fluids in general relativity: velocity potentials and a variational principle, Phys. Rev. D 2 (12) (1970) 2762.
[4] Y.M. Ali, L.C. Zhang, Relativistic heat conduction, Int. J. Heat Mass Transfer 48 (2005) 2397.
[5] V. Mosengeil, Theorie der stationaeren strahlung in einem gleichfoermig bewegten hohlraum, Annalen der Physik 22 (1907) 867.
[6] M. Planck, Zur dynamik bewegter systeme, Ann. der Phys. 331 (1908) 1.
[7] W. Pauli, Theory of Relativity, Pergamon Press, 1958, p. 134.
[8] R.C. Tolman, Relativity, Thermodynamics and Cosmology, Clarendon Press, 1934, p. 152.
[9] H. Ott, Lorentz transformation der warme und der temperature, Zeits. F. Phys 70 (1963) 175.
[10] P.T. Landberg, Does a moving body appear cool, Nature 212 (1966) 571.
[11] T.Q. Qiu, C.L. Tien, Short-pulse laser heating on metals, Int. J. Heat Mass Transfer 35 (1992) 719.
[12] D.D. Joseph, L. Preziosi, Heat waves, Rev. Modern Phys. 61 (1989) 41.
[13] D.D. Joseph, L. Preziosi, Addendum to the paper on heat waves, Rev. Modern Phys. 62 (1990) 375.
[14] M. Chester, Second sound in solid, Phys. Rev. 131 (1963) 2013.
[15] A. Sellitto, V.A. Cimmelli, D. Jou, Analysis of three nonlinear effects in a continuum approach to heat transport in nanosystems, Phys. D 241 (16) (2012) 1344.
[16] B. Kundu, K.S. Lee, Fourier and non-Fourier heat conduction analysis in the absorber plates of a flat-plate solar collector, Solar Energy 86 (10) (2012) 3030.
[17] Q.X. Liu, P.X. Jiang, H. Xiang, Molecular dynamics simulations of non-Fourier heat conduction, Progress Nat. Sci. 18 (8) (2008) 999.
[18] B. Shen, P. Zhang, Notable physical anomalies manifested in non-Fourier heat conduction under the dual-phase-lag model, Int. J. Heat Mass Transfer 51 (2008) 1713.
[19] B. Kundu, K.S. Lee, A non-Fourier analysis for transmitting heat in fins with internal heat generation, Int. J. Heat Mass Transfer 64 (2013) 1153.
[20] E. Marín, On thermal waves, Eur. J. Phys. 34 (2013) L83.
[21] C. Cattaneo, Sur une forme de l'equation de la chaleur elinant le paradoxe d'une propagation instantance, Comptes Rendus de l'Acad. des Sci. 247 (1958) 431.
[22] M.P. Vernotte, Les Paradoxes de la theorie continue de l'equation de la chaleur, Comptes Rendus de l'Acad. des Sci. 246 (1958) 3154.
[23] D.Y. Tzou, Thermal shock phenomena under high-rate response, Ann. Rev. Heat Transfer. 4 (1992) 111.
[24] S.I. Anisinov, B.L. Kapeliovich, T.L. Perel'man, Electron emission from metal surfaces exposed to ultrashort laser pulses, J. Exp. Theoret. Phys. 39 (2) (1974) 375.
[25] R.A. Guyer, J.A. Krumhansl, Solution of the linearized phonon Boltzmann equation, Phys. Rev. 148 (2) (1966) 766.
[26] D.Y. Tzou, A unified field approach for heat-conduction from macro-scales to micro-scales, ASME J. Heat Transfer 117 (1995) 8.
[27] B.D. Coleman, M. Fabrizio, D.R. Owen, On the thermodynamics of second sound in dielectric crystals, Arch. Rational Mech. Anal. 80 (1982) 135.
[28] B.Y. Cao, Z.Y. Guo, Equation of motion of a phonon gas and non-Fourier heat conduction, J. Appl. Phys. 102 (5) (2007) 053503.
[29] B.Y. Cao, Z.Y. Guo, Thermal wave based on the thermomass model, ASME J. Heat Transfer 132 (7) (2010) 072403.
[30] A. Sellitto, V.A. Cimmelli, A continuum approach to thermomass theory, ASME J. Heat Transfer 134 (11) (2012) 112402.
[31] M.K Zhang, B.Y. Cao, Y.C. Guo, Numerical studies on damping of thermal waves, Int. J. Therm. Sci. 84 (2014) 9.
[32] M.K Zhang, B.Y. Cao, Y.C. Guo, Numerical studies on dispersion of thermal waves, Int. J. Heat Mass Transfer 67 (2013) 1072.
[33] V. Bertolaa, E. Cafaro, On the speed of heat, Phys. Lett. A 372 (2007) 1.
[34] V. Bertola, E. Cafaro, Diffusion phenomena and thermodynamics, Phys. Lett. A 374 (2010) 3373.
[35] V. Bertola, E. Cafaro, Heat glides on a curved pathway, Phys. Lett. A 377 (2012) 7.
[36] S. Goldberg, The Lorentz theory of electrons and Einstein's theory of relativity, Am. J. Phys. 37 (1969) 982.
[37] R.G. Newburgh, Relativistic thermodynamics: transformation, invariance and measurement, Nuovo Cimento 52 B (2) (1979) 219.
[38] R.C. Tolman, Relativity, Thermodynamics and Cosmology, Clarendon Press, 1934, p. 152.
[39] R.K. Pathria, Statistical Mechanics, Pergamon Press, 1972, p. 152.
[40] E. Bormashenko, Entropy of relativistic mono-atomic gas and temperature relativistic transformation in thermodynamics, Entropy 9 (3) (2007) 113.
[41] I. Avramov, Relativity and temperature, Russ. J. Phys. Chem. 77 (2003) 179.