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Generalized variational principles for heat conduction models based on Laplace transforms

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ABSTRACT

The classical variational principle does not exist for parabolic and hyperbolic heat conduction equations, which has led to the demand for special variational methods for heat conduction. O'Toole (1967) first used Laplace transforms for the variational principle only for Fourier's law with the first type of boundary condition. In this paper, the Laplace transform strategy is extended to other parabolic and hyperbolic heat conduction models and other types of boundary conditions. Generalized variational principles are given for heat conduction models including Fourier's law, the Cattaneo–Vernotte (CV) model, the Jeffrey model, the two-temperature model and the Guyer–Krumhansl (GK) model, based on Laplace transforms. The Laplace transform method transforms the heat conduction equations of these models into linear variational equations whose variational principles are already known. For the three standard types of boundary conditions, these generalized variational principles are strictly equivalent to the heat conduction equations for these models. The Laplace transform method has stronger convergence in infinite temporal domain problems. In physics, the Laplace transform method is understood as replacing the time dimension with the frequency of the temperature change and the rate of the entropy change.

1. Introduction

Many efforts have been made to formulate variational principles for dissipative processes including mass diffusion and heat conduction. The classical methods were pioneered by Onsager et al. [1–3], Prigogine et al. [4], Biot [5,6], Gyarmati et al. [7,8]. Many other methods have been developed [9–13] with a rather good classification of these methods in the review by Ván and Muschik [14]. These variational principles have been applied to such fields as approximation methods [15,16] and thermal analyses of some structures [17,18]. In dynamics, the classical variational principle generalizes the physical laws or provides methods for solving the differential equations. However, the conditions for the existence of the classical variational principle are so strict that many heat conduction models including Fourier's law do not have classical variational principle model. Therefore, these "variational principles" which go beyond the classical variational principle should be called "generalized variational principles". In nonequilibrium thermodynamics which often involve parabolic equations that do not satisfy the classical variational principle, these generalized variational principles can help us better understand the physical processes, develop better numerical methods and clarify the solution characteristics [14,19].

O'Toole [20] used Laplace transforms to provide variational principles for time-dependent transport processes with Laplace transform [21] $U(x,y,z,p)$ of a function $u(x,y,z,t)$ expressed as $U(x,y,z,p) = \int_0^{+\infty} u(x,y,z,t)e^{-pt}dt$. O'Toole's gave a generalized variational principle for Fourier's law based on Laplace transforms. However, strictly speaking, O'Toole's variational principle for the Fourier heat conduction is only for the first type of boundary condition (Dirichlet), which specifies the boundary temperature, and he did not discuss under what circumstances this Laplace transform method is feasible. However, there are also two other types of the boundary conditions which occur frequently in heat conduction processes, the second (Neumann) and third (Robin) types of boundary conditions. The second type specifies the boundary heat flux while third type specifies both the heat transfer coefficient and the ambient temperature. This work presents generalized variational principles for Fourier heat conduction for all three types of boundary conditions. The physical feasibility is then used to provide a condition which guarantees that the generalized variational principle is strictly equivalent to the Fourier heat conduction equation for various types of the boundary conditions.

The variational problem for non-Fourier heat conduction models, which is related to thermal transport in nanostructures and laser-heating, is also discussed because although Fourier's law of heat conduction accurately describes classical heat conduction problems, it has some limitations [22–25]. Several modified heat

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conduction models have then been proposed such as the Cattaneo–Vernotte (CV) model [26,27], the Jeffrey model [23], the Guyer–Krumhansl (GK) model [28], the two-temperature model [29]. The Laplace transform method also provides generalized variational principles for these non-Fourier heat conduction models for all three types of boundary conditions. The convergence and physical meaning of the Laplace transform method are also discussed.

2. Generalized variational principles based on Laplace transforms

2.1. Generalized variational principle for Fourier's law

Fourier's law and the energy conservation equation are expressed as

$$q + \lambda \nabla T = 0, \tag{1}$$

$$\nabla \cdot q = -\rho c_V \frac{\partial T}{\partial t}. \tag{2}$$

where q is the heat flux, λ is the thermal conductivity, T is the temperature, ρ is the mass density and c_V is the specific heat. The thermal conductivity λ, the specific heat c_V and particularly the mass density ρ are positive and constant in time and space. Eqs. (1) and (2) can be combined to give the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c_V} \nabla^2 T. \tag{3}$$

Applying the Laplace transform $F = \int_0^{+\infty} T e^{-pt} dt$ to Eq. (3) leads to

$$\frac{\lambda}{\rho c_V} \nabla^2 F = pF - T|_{t=0}. \tag{4}$$

Eq. (4) is a linear variational equation whose variational principle is already known [30–32]. For the first type of boundary condition, the given boundary temperature means that the Laplace transform of the boundary temperature is also given. Therefore, the first type of boundary condition for the temperature is also the first type of boundary condition for F. The generalized variational principle for the first type of boundary condition is:

$$\delta \left\{ \iiint_D \left[\frac{\lambda}{\rho c_V} |\nabla F|^2 + pF^2 - 2F(T|_{t=0}) \right] dV \right\} = 0, \tag{5}$$

where D is the spatial domain. The second and third types of boundary conditions can be expressed as

$$\left. \left(-q \cdot \vec{n} + hT \right) \right|_{\Gamma} = \sigma, \tag{6}$$

where Γ is the boundary of D, h is the heat transfer coefficient and σ is a constant. Substituting Fourier's law into Eq. (6) leads to

$$\left. \left(\lambda \frac{\partial T}{\partial n} + hT \right) \right|_{\Gamma} = \sigma. \tag{7}$$

The Laplace transform of Eq. (7) is

$$\left. \left(\lambda \frac{\partial F}{\partial n} + hF \right) \right|_{\Gamma} = \frac{\sigma}{p}. \tag{8}$$

The second and third types of boundary conditions for the temperature are then also the second and third types of boundary problems for F. Therefore, the generalized variational principle for the second and third types of boundary conditions are

$$\delta \left\{ \iiint_D \left[\lambda |\nabla F|^2 + \rho c_V p F^2 - 2\rho c_V F(T|_{t=0}) \right] dV + \int_{\Gamma} \left(hF^2 - 2\frac{F\sigma}{p} \right) dS \right\} = 0. \tag{9}$$

2.2. Engineering equivalence of the generalized variational principle

These variational principles are shown here to be equivalent to the heat conduction equation based on the physical feasibility of the boundary conditions. In engineering heat conduction problem which is physically feasible, the boundary and initial temperatures must be finite. In addition, the heat conduction equation, Eq. (3), is a parabolic equation whose maximum principle guarantees that the maximum values of the temperature field must appear in the boundary or initial conditions. Therefore, the whole temperature field must also be finite, which guarantee the integral $F = \int_0^{+\infty} T e^{-pt} dt$ to be convergent. Besides the temperature, the heat transfer coefficient, h, and σ must also be finite. Thus, for physically feasible heat conduction problems, the Laplace transforms of the temperature field and boundary conditions must exist, which shows the feasibility of Eqs. (4) and (8). In addition, Eqs. (4) and (8) are all linear variational types which leads to the equivalence for the variational principles expressed by Eqs. (5) and (9). Thus these variational principles can determine the existence and uniqueness of F. These variational principles can then be shown to be strictly equivalent to Eq. (3). First, T must be continuous in time because the heat conduction equation has a time differential term $\frac{\partial T}{\partial t}$. The convergence of the Laplace transform and the continuity of T guarantees one-to-one correspondence between T and F from Lerch's Theorem [21]. Therefore, the variational principles expressed by Eqs. (5) and (9) determine not only the existence and uniqueness of F but also determine the existence and uniqueness of T. In summary, for various finite boundary conditions in engineering problems, the generalized variational principles expressed by Eqs. (5) and (9) are equivalent to Eq. (3).

2.3. Generalized variational principle for the CV model

The CV model [26,27] is expressed as

$$q + \tau \frac{\partial q}{\partial t} + \lambda \nabla T = 0, \tag{10}$$

where τ is the thermal relaxation time. The heat conduction equation of the CV model is

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = \frac{\lambda}{\rho c_V} \nabla^2 T. \tag{11}$$

Taking the Laplace transform of Eq. (11) leads to

$$\frac{\lambda}{\rho c_V} \nabla^2 F - (p + \tau p^2) F + (\tau p + 1) T|_{t=0} + \tau \left. \frac{\partial T}{\partial t} \right|_{t=0} = 0. \tag{12}$$

Eq. (12) is also a linear variational equation. Similar to Fourier's law, the first type of boundary condition for the temperature for Eq. (12) is also the first type of boundary condition for F. Therefore, the generalized variational principle for the first type of boundary condition is

$$\delta \left\{ \iiint_D \left\{ \frac{\lambda}{\rho c_V} |\nabla F|^2 + (p + \tau p^2) F^2 - 2\left[(\tau p + 1) T|_{t=0} + \tau \left. \frac{\partial T}{\partial t} \right|_{t=0} \right] F \right\} dV \right\} = 0. \tag{13}$$

For the second and third types of boundary conditions, substituting the CV model into Eq. (6) leads to

$$\left[\left(\lambda\nabla T+\tau\frac{\partial q}{\partial t}\right)\cdot\vec{n}+hT\right]\Big|_{\Gamma}=\sigma. \tag{14}$$

The Laplace transforms of Eqs. (14) and (10) are

$$\left[(\lambda\nabla F+p\tau Q-\tau q|_{t=0})\cdot\vec{n}+hF\right]\Big|_{\Gamma}=\frac{\sigma}{p}, \tag{15}$$

$$Q+p\tau Q-\tau q|_{t=0}+\lambda\nabla F=0, \tag{16}$$

where $Q=\int_0^{+\infty}qe^{-pt}dt$. From Eq. (16),

$$Q=-\frac{\lambda\nabla F-\tau q|_{t=0}}{1+p\tau}. \tag{17}$$

Substituting Eq. (17) into Eq. (15) leads to

$$\left[\frac{\lambda}{1+p\tau}\frac{\partial F}{\partial n}-\frac{\tau}{1+p\tau}\left(q|_{t=0}\cdot\vec{n}\right)+hF\right]\Big|_{\Gamma}=\frac{\sigma}{p} \tag{18}$$

From Eq. (18), the second and third types of boundary conditions for the temperature are also the second and third types of boundary conditions for F. Then, the generalized variational principle for the second and third types of boundary conditions is:

$$\delta\left\{\iiint_D\left\{\lambda|\nabla F|^2+\rho c_V(p+\tau p^2)F^2-2\rho c_V\left[(\tau p+1)T|_{t=0}+\tau\frac{\partial T}{\partial t}\Big|_{t=0}\right]F\right\}dV+\int_\Gamma\left\{(\tau p+1)hF^2-2\frac{F\sigma(\tau p+1)}{p}-2F\tau\left(q|_{t=0}\cdot\vec{n}\right)\right\}dS=0. \tag{19}$$

However, $(q|_{t=0}\cdot\vec{n})$ is still unknown but can be expressed from the boundary and initial conditions. Equation (6) at $t=0$ gives

$$\left(-q|_{t=0}\cdot\vec{n}+hT|_{t=0}\right)\Big|_{\Gamma}=\sigma. \tag{20}$$

Substituting Eq. (20) into Eq. (19) leads to

$$\delta\left\{\iiint_D\left\{\lambda|\nabla F|^2+\rho c_V(p+\tau p^2)F^2-2\rho c_V\left[(\tau p+1)T|_{t=0}+\tau\frac{\partial T}{\partial t}\Big|_{t=0}\right]F\right\}dV+\int_\Gamma\left\{(\tau p+1)hF^2-2\frac{F\sigma(\tau p+1)}{p}-2F\tau(hT|_{t=0}-\sigma)\right\}dS\right\}=0, \tag{21}$$

which is the final generalized variational principle for the second and third types of boundary conditions.

2.4. Generalized variational principle for the Jeffrey model

The Jeffrey model [23] is expressed as

$$q+\tau\frac{\partial q}{\partial t}+k\nabla T+k_F\tau\frac{\partial}{\partial t}(\nabla T)=0, \tag{22}$$

where k_F is the thermal conductivity for Fourier heat conduction and k is the total thermal conductivity. The heat conduction equation for the Jeffrey model is

$$\frac{1}{\tau}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=\frac{k}{\rho c_V\tau}\nabla^2 T+\frac{k_F}{\rho c_V}\frac{\partial}{\partial t}(\nabla^2 T). \tag{23}$$

The Laplace transform of Eq. (23) is

$$\left(\frac{k}{\rho c_V\tau}+\frac{pk_F}{\rho c_V}\right)\nabla^2 F-\left(p^2+\frac{p}{\tau}\right)F+\left(p+\frac{1}{\tau}\right)T|_{t=0}+\frac{\partial T}{\partial t}\Big|_{t=0}-\frac{k_F}{\rho c_V}\nabla^2 T\Big|_{t=0}=0, \tag{24}$$

which is still a linear variational equation. As with the other cases, the first type of boundary condition for the temperature is also the first type of boundary condition for F. Therefore, the generalized variational principle for the first type of boundary condition is

$$\delta\left\{\iiint_D\left\{\begin{array}{l}\left(\frac{k}{\rho c_V\tau}+\frac{pk_F}{\rho c_V}\right)|\nabla F|^2+(p^2+\frac{p}{\tau})F^2-\\ 2\left[(p+\frac{1}{\tau})T|_{t=0}+\frac{\partial T}{\partial t}|_{t=0}-\frac{k_F}{\rho c_V}\nabla^2 T\Big|_{t=0}\right]F\end{array}\right\}dV\right\}=0 \tag{25}$$

The method in **Section** 2.3 is again used for the second and third types of boundary conditions. Substituting Eq. (22) into Eq. (6) leads to

$$\left[\left(k\nabla T+\tau\frac{\partial q}{\partial t}+k_F\tau\frac{\partial}{\partial t}(\nabla T)\right)\cdot\vec{n}+hT\right]\Big|_{\Gamma}=\sigma. \tag{26}$$

The Laplace transforms of Eqs. (26) and (22) are

$$\left[(k\nabla F+p\tau Q-\tau q|_{t=0}+k_Fp\tau\nabla F-k_F\tau\nabla T|_{t=0})\cdot\vec{n}+hF\right]\Big|_{\Gamma}=\frac{\sigma}{p}, \tag{27}$$

$$Q+p\tau Q-\tau q|_{t=0}+k\nabla F+k_Fp\tau\nabla F-k_F\tau\nabla T|_{t=0}=0. \tag{28}$$

From Eq. (28),

$$Q=-\frac{k\nabla F+k_Fp\tau\nabla F-k_F\tau\nabla T|_{t=0}-\tau q|_{t=0}}{1+p\tau}. \tag{29}$$

Substituting Eq. (29) into Eq. (27) leads to

$$\left[\left(\frac{k+k_Fp\tau}{1+p\tau}\right)\frac{\partial F}{\partial n}-\frac{\tau}{1+p\tau}\left[(k_F\nabla T|_{t=0}+q|_{t=0})\cdot\vec{n}\right]+hF\right]\Big|_{\Gamma}=\frac{\sigma}{p}, \tag{30}$$

which are also the second and third types of boundary conditions for F. Then, the generalized variational principle for the second and third types of boundary conditions is

$$\delta\left\{\iiint_D\left\{(k+p\tau k_F)|\nabla F|^2+\rho c_V\tau\left(p^2+\frac{p}{\tau}\right)F^2-2\rho c_V\tau F\left[(p+\frac{1}{\tau})T\Big|_{t=0}+\frac{\partial T}{\partial t}\Big|_{t=0}-\frac{k_F}{\rho c_V}\nabla^2 T\Big|_{t=0}\right]\right\}dV+\int_\Gamma\left\{(\tau p+1)hF^2-2\frac{F\sigma(\tau p+1)}{p}-2F\tau\left[(k_F\nabla T|_{t=0}+q|_{t=0})\cdot\vec{n}\right]\right\}dS\right\}=0. \tag{31}$$

Eq. (20) still holds and the final generalized variational principle is

$$\delta\left\{\iiint_D\left\{(k+p\tau k_F)|\nabla F|^2+\rho c_V\tau\left(p^2+\frac{p}{\tau}\right)F^2-2\rho c_V\tau F\left[\left(p+\frac{1}{\tau}\right)T\Big|_{t=0}+\frac{\partial T}{\partial t}\Big|_{t=0}-\frac{k_F}{\rho c_V}\nabla^2 T\Big|_{t=0}\right]\right\}dV+\int_\Gamma\left\{(\tau p+1)hF^2-2\frac{F\sigma(\tau p+1)}{p}-2F\tau\left[(k_F\nabla T|_{t=0})\cdot\vec{n}+hT|_{t=0}-\sigma\right]\right\}dS\right\}=0. \tag{32}$$

2.5. Generalized variational principles for other models

This method can also be used to derive generalized variational principles for other heat conduction models. This section provides generalized variational principles for two more complex models using the Laplace transform method.

Anisinov et al. [29] proposed the two-temperature model for metals with the heat conduction equation expressed as

$$\nabla^2 T_e+\frac{\alpha_e}{C_E^2}\frac{\partial}{\partial t}(\nabla^2 T_e)=\frac{1}{\alpha_E}\frac{\partial T_e}{\partial t}+\frac{1}{C_E^2}\frac{\partial^2 T_e}{\partial t^2}, \tag{33}$$

where T_e is the electron temperature, α_E is the equivalent thermal diffusivity, α_e is the thermal diffusivity of the electrons and C_E is the heat wave velocity. The Laplace transform of Eq. (33) is

$$\left(1+\frac{p\alpha_e}{C_E^2}\right)\nabla^2F_e-\left(\frac{p}{\alpha_E}+\frac{p^2}{C_E^2}\right)F_e+\frac{1}{C_E^2}\left(p\nabla T_e|_{t=0}+\left.\frac{\partial T_e}{\partial t}\right|_{t=0}\right)+\frac{1}{\alpha_E}\nabla T_e|_{t=0}-\frac{\alpha_e}{C_E^2}\nabla^2T_e\Big|_{t=0}=0. \tag{34}$$

For the first type of boundary condition, the generalized variational principle is

$$\delta\left\{\iiint_D\left\{\left(1+\frac{p\alpha_e}{C_E^2}\right)|\nabla F_e|^2+\left(\frac{p}{\alpha_E}+\frac{p^2}{C_E^2}\right)F_e^2-2\left[\frac{1}{C_E^2}\left(p\nabla T_e|_{t=0}+\left.\frac{\partial T_e}{\partial t}\right|_{t=0}\right)+\frac{1}{\alpha_E}\nabla T_e|_{t=0}-\frac{\alpha_e}{C_E^2}\nabla^2T_e\Big|_{t=0}\right]F_e\right\}dV\right\}=0. \tag{35}$$

For the second and third types of boundary conditions, the methodology is the same as in **Sections** 2.3 and 2.4. The final generalized variational principle is

$$\delta\left\{\iiint_D\left\{\left(1+\frac{p\alpha_e}{C_E^2}\right)|\nabla F_e|^2+\left(\frac{p}{\alpha_E}+\frac{p^2}{C_E^2}\right)F_e^2-2F_e\left[\frac{1}{C_E^2}\left(p\nabla T_e|_{t=0}+\left.\frac{\partial T_e}{\partial t}\right|_{t=0}\right)+\frac{1}{\alpha_E}\nabla T_e|_{t=0}-\frac{\alpha_e}{C_E^2}\nabla^2T_e\Big|_{t=0}\right]\right\}dV+\int_\Gamma\left\{\frac{h}{k}\left(1+\frac{p\alpha_e}{C_E^2}\right)F_e^2-\frac{2F\sigma}{pk}\left(1+\frac{p\alpha_e}{C_E^2}\right)\right\}dS\right\}=0. \tag{36}$$

The GK model [28] is a classical model for phonon heat conduction whose heat conduction equation is

$$\nabla^2T+\frac{9\tau_N}{5}\frac{\partial}{\partial t}(\nabla^2T)=\frac{2}{\tau_Rc^2}\frac{\partial T}{\partial t}+\frac{3}{c^2}\frac{\partial^2T}{\partial t^2}, \tag{37}$$

where τ_N is the single-phonon relaxation time for normal processes, τ_R is the momentum loss relaxation time and c is the isothermal first-sound velocity. It is worth mentioning that variational problems about the GK model have been discussed from different views [33–34]. The Laplace transform of Eq. (37) is

$$\left(1+\frac{9\tau_N}{5}\right)\nabla^2F-\left(\frac{2p}{\tau_Rc^2}+\frac{3p^2}{c^2}\right)F-\frac{9\tau_N}{5}\nabla^2T\Big|_{t=0}+\left(\frac{2}{\tau_Rc^2}+\frac{3p}{c^2}\right)T\Big|_{t=0}+\frac{3}{c^2}\left.\frac{\partial T}{\partial t}\right|_{t=0}=0. \tag{38}$$

For the first type of boundary condition, the generalized variational principle is

$$\delta\left\{\iiint_D\left\{\left(1+\frac{9\tau_N}{5}\right)|\nabla F_e|^2+\left(\frac{2p}{\tau_Rc^2}+\frac{3p^2}{c^2}\right)F^2-2\left[\left(\frac{2}{\tau_Rc^2}+\frac{3p}{c^2}\right)T\Big|_{t=0}+\frac{3}{c^2}\left.\frac{\partial T}{\partial t}\right|_{t=0}-\frac{9\tau_N}{5}\nabla^2T\Big|_{t=0}\right]F\right\}dV\right\}=0. \tag{39}$$

For the second and third types of boundary conditions, the methodology is the same as in **Sections** 2.3 and 2.4. The final generalized variational principle is

$$\delta\left\{\iiint_D\{\left(1+\frac{9\tau_N}{5}\right)|\nabla F_e|^2+\left(\frac{2p}{\tau_Rc^2}+\frac{3p^2}{c^2}\right)F^2-2\left[\left(\frac{2}{\tau_Rc^2}+\frac{3p}{c^2}\right)T\Big|_{t=0}+\frac{3}{c^2}\left.\frac{\partial T}{\partial t}\right|_{t=0}-\frac{9\tau_N}{5}\nabla^2T\Big|_{t=0}\right]F\}dV+\int_\Gamma\left\{\frac{(1+\frac{9\tau_N}{5})}{\left(\frac{3\tau_Npc^2C_p}{5}-\frac{c^2C_p}{3}\right)}\left[(p+\frac{1}{\tau_R})hF^2-2F\left(2\sigma+\frac{\sigma}{\tau_Rp}-hT|_{t=0}+\frac{3\tau_Nc^2C_p}{5}\nabla T|_{t=0}\cdot\vec{n}\right)\right]\right\}dS\right\}=0 \tag{40}$$

3. Characteristic of the Laplace transform method

3.1. Convergence

Generally speaking, the temporal domain of the heat conduction problem is infinite which requires the convergence of the integral in the temporal domain. The Laplace transform method provide stronger convergence for heat conduction variational principles because a convergence factor, e^{-pt}, is added by the Laplace transform. For example, the heat conduction equation for steady Fourier heat conduction is

$$\nabla(\lambda\nabla T)=0. \tag{41}$$

The variational principle of Eq. (41) is

$$\delta\left\{\iiint_D\left[\lambda|\nabla T|^2\right]dV\right\}=0 \tag{42}$$

For Eq. (42), the integral in the temporal domain is

$$\int_0^{+\infty}dt\left\{\iiint_D[\lambda|\nabla T|^2]dV\right\}=\lim_{t\to+\infty}\left\{t\iiint_D\left[\lambda|\nabla T|^2\right]dV\right\}=+\infty \tag{43}$$

which is not convergent. However, with the Laplace transform method, the integral of the steady variational principle is

$$\iiint_D\left[\frac{\lambda}{\rho c_V}|\nabla F|^2+pF^2-2F(T|_{t=0})\right]dV. \tag{44}$$

For steady-state problems,

$$F=\int_0^{+\infty}Te^{-pt}dt=\frac{T}{p}. \tag{45}$$

Substituting Eq. (45) into Eq. (44) leads to

$$\iiint_D\left[\frac{\lambda}{\rho c_Vp^2}|\nabla F|^2+\frac{T^2}{p}-2\frac{T}{p}(T|_{t=0})\right]dV, \tag{46}$$

which is convergent.

3.2. Physical meaning

The Laplace transform separates the temperature fields into components with different frequencies, Imp, and different growth rates in exponent Rep, where Rep is the real part of p and Imp is the imaginary part of p. The physical meaning of Imp is clearly, the frequency of the temperature variation. Rep shows the temperature growth rate as $e^{\mathrm{Re}(p)}$ and has a further physical meaning. For pure heat conduction problems, the entropy per unit volume, S, can be calculated as

$$TdS=\rho c_VdT. \tag{47}$$

From Eq. (47),

$$T=C_1e^{\frac{S}{\rho c_V}}, \tag{48}$$

where C_1 is a constant. Therefore, the growth rate given by the exponential is expressed as $\frac{1}{\rho c_V}\frac{dS}{dt}$. So, the Laplace transform also breaks the temperature field into components with different rates of entropy change. The Laplace transform also transforms $T(x,y,z,t)$ into $F(x,y,z,p)$. Thus, the Laplace transform method replaces the time dimension with the frequency and the rate of entropy change. In other fields, e.g. signaling system, the Laplace transform, where Re$p\neq0$, is usually considered as an improvement of Fourier transformation where Re$p=0$. When Re$p<0$, the

Laplace transform could have a stronger convergence than Fourier transformation's. Therefore, the effect of Re*p*, which is added by the Laplace transform, is considered to improve the convergence for Fourier transformation. What's more, Im*p* is considered to have a very important engineering and physical meaning, that is the frequency spectrum, while Re*p* is considered as a mathematical method for improving convergence. However, from the above results, we find that for heat conduction problems, Re*p* is not only a mathematical method but also the rate of entropy change spectrum, which is also very important and meaningful. Different from other fields, for heat conduction problems, both Re*p* and Im*p* have the same important position.

4. Conclusions

In this paper, O'Toole's idea of using Laplace transforms to obtain generalized variational principles is extended to other parabolic and hyperbolic heat conduction models and other types of the boundary conditions. This Laplace transform method has the following characteristics.

(1) The Laplace transform method turns the parabolic and hyperbolic heat conduction equations for various models into linear variational equations with known variational principles. Generalized variational principles are then obtained for Fourier's law, the CV model, the Jeffrey model, the two-temperature model and the GK model.
(2) The generalized variational principles given by the Laplace transform method are equivalent to the heat conduction equation with finite boundary conditions. In addition, for the three common types of boundary conditions, the Laplace transform does not change the type of the boundary condition which also simplifies the variational problems.
(3) The Laplace transform method provides stronger convergence for the heat conduction variational principles because the convergence factor e^{-pt} is added by the Laplace transform. The physical meaning of this method is demonstrated to replace the time dimension with the frequency of the temperature change and the rate of entropy change by transforming $T(x,y,z,t)$ into $F(x,y,z,p)$.

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